বিদ্যাসাগর বিশ্ববিদ্যালয় VIDYASAGAR UNIVERSITY

## Question Paper

## B.Sc. Honours Examinations 2020

(Under CBCS Pattern)
Semester - III
Subject: MATHEAMATICS
Paper: C5T
(Theory of Real Functions and Introduction to Metric Space)

## Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable. The figures in the margin indicate full marks.

Answer any three from the following questions :

1. (a) Examine with reason whether $\lim _{x \rightarrow 0}\left(\sin \frac{1}{x}+x \sin \frac{1}{x}\right)$ exist or not.
(b) Give examples of a function which is
(i) Continues and bounded on $\mathbb{R}$, attains its suprimum but not infimum.
(ii) Continues and bounded on $\mathbb{R}$, attains its infimum but not its suprimum.
(iii) Continunes and bounded on an interva, but attains neither its suprimum nor infimum.
(c) Let $[a, b]$ be a closed and bounded interval and $f:[a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. If $f(a)$ and $f(b)$ are opposite sign then show that there exists at least a point c in the open interval $(a, b)$ such that $f(c)=0$.
(d) Does Rolle's theorem hold for $f(x)=1-|1-x|$ in $[0,2]$ Justify.
(e) Let $f(x)=\left\{\begin{array}{ll}x\left\{1+\frac{1}{3}\left(\log x^{2}\right),\right. & x \neq 0 \\ 0 & , \quad x=0\end{array}\right.$ Show that $f$ is continuous at $x=0$ but not derivable there.
2. (a) Show that the Dirichlet's function is everywhere discountnuous on $\mathbb{R}$.
(b) Let $[a, b]$ be a closed and bouded interval and a function $\mathbb{R}$ be continuous on $[a, b]$. If $f(a) \neq f(b)$ then fattains every value between $f(a)$ and $f(b)$ at least once in $(a, b)$. Is the converse ture ? Justify.
(c) Give and example of a function $f$ defined on an interval $I$ such that
(i) $f$ has jump discontinuity at a point of $I$.
(ii) $f$ has removable discontinuity at a point of $I$.
(iii) $f$ has infinite discontinuity at a point of I.4.
(d) (i) Prove that for no real value of $k$, the equation $x^{3}-12 x+k=0$ has two real roots in [-1,.1].
(ii) Prove that there does not exist a function $\varphi$ such that $\varphi^{-}(x)=f(x)$ on $[0,2]$ where $f(x)=x-[x]$.
(e) Prove that $x-\frac{x^{3}}{x}<\sin x<-\frac{x^{3}}{6}+\frac{x^{5}}{120}$ for all $x>0$.
3. (a) Prove that there exists $x \in\left(0, \frac{\pi}{2}\right)$ such that $x=\cos x$.
(b) Let $D \subset \mathbb{R}$ and a function $f: D \rightarrow \mathbb{R}$ be uniformly continuous on $D$. If $\left\{x_{n}\right\}$ be a Cauchy sequence in $D$ then show that $\left\{f\left(x_{n}\right)\right\}$ is a Cauchy sequence in $\mathbb{R}$. If we drop the condition "uniformity", then is the above reseult hold ? Justify.
(c) If $f(x)$ be differentiable at $x=a$ show that
$\lim _{x \rightarrow a} \frac{(x+a) f(x)-2 a f(a)}{x-a}=f(a)+2 a f^{\prime}(a)$.
(d) (i) State Rolle's theorem. Is the set of conditions of Rolle's theorem a necessary condition? Justify.
(ii) If a function $f$ is continuous at a point $x=0$, prove t hat $x f(x)$ is derivable at $x=0.5$
(e) State and prove Lagrange's mean value theorem. Give its geometrical signifincance.
4. (a) $f:[0,1] \rightarrow \mathbb{R}$ is continuous on $[0,1]$ and $f$ assumes only rational values. If $f\left(\frac{1}{2}\right)=\frac{1}{2}$, prove that $f(x)=\frac{1}{2}$ for all $x \in[0,1]$.
(b) (i) Give an examples to show that a function which is continuous on an open bouded interval may not be uniformly continuous there.
(ii) Let f be continuous on $[\mathrm{a}, \mathrm{b}]$ and $f(x)=0$ when $x$ is rational. Show that $f(x)=0$ for every $x \in[a, b]$.
(c) Find $f^{\prime}(0)$ [if exist] for the function $f(x)=\left\{\begin{array}{l}3+2 x,-\frac{3}{2}<x \leq 0 \\ 3-2 x, 0<x<\frac{3}{2}\end{array}\right.$
(d) Prove that between any two real roots of $e^{x} \sin x=1$, there exist at least one real root of $e^{x} \cos x+1=0$.
(e) Expand $\sin x, x \in \mathbb{R}$, in powers of $x$ by Tailor's series expansion.
(f) Find the minimum value (if exist) of the function defined by $f(x)=x^{x},(x>0)$
(g) Show that the greatest value of $x^{m} y^{n},(x>0, y>0)$ and $x+y=k(k=$ constant $)$ is $\frac{m^{n} n^{n} k^{m+n}}{(m+n)^{n}}$.
5. (a) Prove that between any two real roots of $e^{x} \sin x=1$ there exist at least one real root of $e^{x} \cos x+1=0$.
(b) A function $f$ is thrice differentiable on $[a, b]$ and $f(a)=f(b)=0$ and also $f^{\prime}(a)=f^{\prime}(b)=0$. Prove that the second derivative of $f$ vanishes at c , where $a<c<b$.
(c) Define discrete and pseudo metric space.
(d) On the real line $\mathbb{R}$, show that a singleton set is not an open set.
(e) Let $X$ be the set of all sequences of real numbers containing only a finite number of non-zero element. Let $d: X \times X \rightarrow X$ be defined by $d\left(\left\{x_{n}\right\},\left\{y_{n}\right\}\right)=\left\{\sum_{r=1}^{\infty}\left(x_{r}-y_{r}\right)^{2}\right\}^{\frac{1}{2}}$.
(f) Give an example to show that the continuous image of an open bounded interval may not be an open bouded interval.
6. (a) In the mean value theorem $f(x+h)=f(x)+h f^{\prime}(x+\theta h), 0<\theta<1$, prove that

$$
\begin{equation*}
\lim _{h \rightarrow 0+} \theta=\frac{1}{2} \text { if } f(x)=\sin x . \tag{4}
\end{equation*}
$$

(b) Show that any discrete metric space is a complete metric space.
(c) Show that in any metric space, a finite set has no limit point.
(d) Show by example that in any metric space, the Cantor intersection theorem may not hold good if any of the following conditions is not satisfied:
(i) $\left\{F_{n}\right\}$ is a sequence of closed sets.
(ii) $\delta\left(F_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$ where $\delta(A)$ denotes the diameter of the set $A$.
(e) We know in a metric space $(X, d)$, "the union of a finite number of closed sets is closed". In this result if we drop the finiteness, then is the result hold good? Justify.

(iii) Give and example of a group $G$ and one of its subgroups $H$ such that $a H=b H$ but $H a \neq H b$ for some $a, b \in G$
(iv) Show that there does not exist any isomorphism between the additive group of all real numbers $(\mathbb{R},+)$ and the multiplicative group of all non-zero real numbers $\left(\mathbb{R}^{*}, \cdot\right)$.
(v) If $(G, 0)$ be group and $a \in G$, prove that any conjugate of the element $a$ has the same order as that of $a$. $2 \times 5$
(b) (i) Prove that a group $\left(G,{ }^{*}\right)$ is commutative if and only if $(a * b)^{n}=a^{n} * b^{n}$ for any three consecutive integers $n$ and for all $a, b \in G$.
(ii) (a) Show that $H=\left\{e,\left(\begin{array}{ll}1 & 2\end{array}\right)\left(\begin{array}{ll}3 & 4\end{array}\right)\right\}$ is not a normal subgroup of $S_{4}$.
(b) Find all homomorphisms of the group $(\mathbb{Z},+)$ to itself. $2+3=5$
2. (a) (i) Give example of two subgroups $H, K$ such that their product $H K$ is not a subgroup.
(ii) Let $G$ be a group and $S$ be a non-empty subset of $G$. Define normalizer of $S$ in $G$. Then show that the normalizer of $S$ is a subgroup of $G$.
(iii) Suppose $N$ is a normal subgroup of a group $G$. If $G$ is an abelian group then $G / N$ is a cyclic group - is this statement true ? Give logic in support of your answer.
(v) Show that $G L(2, \mathbb{R}) / S L(2, \mathbb{R})$ is isomorphic with $\left(\mathbb{R}^{*}, \cdot\right) .2+(1+2)+2+3=10$
(b) (i) Let $S$ be a non-empty subset of a group $G$. If $\langle S\rangle$ denotes the subgroup of $G$ generated by $S$, then prove that
$<S>=\left\{\prod_{i=1}^{n} s_{i}^{e_{i}} \mid s_{i} \in S, e_{i}= \pm 1, i=1,2, \ldots, n ; n \in \mathbb{N}\right\}$.
(ii) Using First Isomorphism theorem show that $\mathbb{Z}_{9}$ is not a homomorphic image of $\mathbb{Z}_{16}$.
(iii) Prove that $K=\left\{e,\left(\begin{array}{ll}1 & 2\end{array}\right)\left(\begin{array}{ll}3 & 4\end{array}\right),\left(\begin{array}{ll}1 & 3\end{array}\right)\left(\begin{array}{ll}2 & 4\end{array}\right),\left(\begin{array}{ll}1 & 4\end{array}\right)\left(\begin{array}{ll}2 & 3\end{array}\right)\right\}$ is a normal subgroup of $A_{4}$.
3. (a) (i) Prove that $H=\left\{\left.\left(\begin{array}{cc}a & b \\ -b & a\end{array}\right) \in G L(2, \mathbb{R}) \right\rvert\,\right.$ either $a \neq 0$ or $\left.b \neq 0\right\}$ is a subgroup of $G L(2, \mathbb{R})$.
(ii) Prove that any finitely generated subgroup of $(\mathbb{Q},+)$ is cyclic.
(iii) Prove that every group of order 49 contains a subgroup of order 7 .
(iv) Let $G$ be a group. Prove that the mapping $f: G \rightarrow G$ defined by $f(a)=a^{-1}$ is a homomorphism if and only if $G$ is commutative. $\quad 3+3+2+2=10$
(b) (i) Let $G$ be a group such that $|G|<320$. Suppose $H, K$ be two subgroups of $G$ such that $|H|=35,|K|=45$. The find the order of $G$.
(ii) Let $G$ be a group. Prove that if $G / Z(G)$ is cyclic, then $G$ is abelian.
(iii) Let K be a normal subgroup of a group G such that $[G: K]=m$. If $n$ is a positive integer such that $\operatorname{gcd}(m, n)=1$, then show that $K \supseteq\{g \in G \mid o(g)=n\}$.
(iv) Prove that up to isomorphism there are only two groups of order 4.

$$
2+2+2+4=10
$$

4. (a) (i) Prove that every group of order $p^{2}$ is abelian, where $p$ is prime.
(ii) Let $G$ be cyclic group and $H$ be a subgroup of $G$. Prove that factor group $\frac{G}{H}$ is cyclic. Is the converse true? Justify.
(b) (i) State the converse of Lagrange's theorem for finite group and justify with an example whether the converse of Lagrange's theorem is true or false.
(ii) Show that the (external) direct product $\mathbb{Z}_{3} \times \mathbb{Z}_{4}$ of the cyclic group $\mathbb{Z}_{3}$ and $\mathbb{Z}_{4}$ is a clyclic group.
$1+5+4$
5. (a) Two finite cyclic groups of the same order are isomorphic.
(b) (i) Let $G$ be a group of order 10 and $G^{\prime}$ be a group of order 6. Prove that there does not exists a homomorphism of $G$ onto $G^{\prime}$.
(ii) Let $G=\left(\mathbb{Z}_{5},+\right)$ and $\varphi: G \rightarrow G$ be defined by $\varphi(\bar{x})=3 \bar{x}, \bar{x} \in \mathbb{Z}_{5}$. Find $\operatorname{ker} \varphi$.
6. (a) (i) Prove that every proper subgroup of a symmetric group $S_{3}$ is cyclic.
(ii) Let $H$ be a subgroup of a group $G$ and, $b \in G$. Prove that $a H \cap b H=\varphi$ if and only if $b \notin a H$.
(iii) Let $P$ and $Q$ be subgroups of a group $G$ and $\operatorname{gcd}(0(P), 0(Q))=1$. Prove that $P \cap Q=\left\{e_{G}\right\}$. $4+3+3$
(b) (i) Prove that the order of every subgroup of a finite group $G$ is a divisor of the order of $G$.
(ii) Prove that a non-commutative group of a order $2 n$, where $n$ is odd prime, must have a subgroup of order $n$.
(iii) Prove that a group of order 27 must have a subgroup of order 3 .

| বিদ্যাসাগর বিশ্ববিদ্যালয় VIDYASAGAR UNIVERSITY <br> Question Paper |  |  |
| :---: | :---: | :---: |
| B.Sc. Honours Examinations 20 <br> (Under CBCS Pattern) <br> Semester - III <br> Subject: MATHEAMATICS <br> Paper: C7T |  |  |
| Full Marks: 60 Time : 3 Hours |  |  |
| Candiates are required to give their answer in their own words as far as practicable. <br> The figures in the margin indicate full marks. |  |  |
| THEORY [Marks 40] <br> Answer any $\boldsymbol{t w o}$ from the following questions: <br> 1. (a) Explain Newton-Raphson method to solve the equation $g(x)=0$. <br> (b) Find the rate of convergence of Newton-Raphson method. <br> (c) Find a real root of the equation $f(x) \equiv x^{3}-2 x-5=0$ lies between 2 and 3 by Resula-Falsi method. <br> 2. (a) Discuss Gauss-elimination method to slove the system of linear equation. |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

(b) Solve the folloiwng equation by Gauss-elimination method.

$$
\begin{aligned}
& 2 x_{1}+x_{2}+x_{3}=4 \\
& x_{1}-x_{2}+2 x_{3}=2 \\
& 2 x_{1}+2 x_{2}-x_{3}=3
\end{aligned}
$$

(c) State the differences between direct and iterative methods.
3. (a) Find an LU- decomposition of the matrix $A=\left[\begin{array}{ccc}2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0\end{array}\right]$ and use it to solve the
system $A x=\left[\begin{array}{c}-3 \\ -12 \\ 6\end{array}\right]$ where $x=\left[\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right]$.
(b) Deduce Lagrange interpolation method.
4. (a) Describe Euler's method and modified Enler's method to solve the following differential equation

$$
\frac{d y}{d x}=f(x, y), \quad y\left(x_{0}\right)=y_{0}
$$

(b) Given $\frac{d y}{d x}=x^{2}+y^{2}$, and when $x=0, y=1$. Find the values of $y(0.1)$ by fourth order Runge-Kutta method.

## PRACTICAL [Marks 20]

## Group - A

Answer any one from the following questions :

Each question carries 10 marks.

1. Write a program to evaluate $\int_{12}^{3}(x \log 2 x+\sin 2 x) d x$ by trapezoidal rule taking 140 subintervals.
2. Write a program to find the value of $y(0.1)$ from the differential equation $\frac{d y}{d x}=x^{2}+y, y(0.1)=1$.
3. Write a program to find the sum of the following series $1+\frac{1}{2}+\frac{1}{3} \ldots+\frac{1}{10050}$.
4. Write a program to find a root of the equation $x^{3}-2 x-1=0$ by bisection method.
5. Write a program to solve the equation $2 x-\sin x-1=0$ using fixed point iteration method.
6. Write a program to find a real root of $x^{5}+3 x^{2}-1=0$ by Newton-Raphosn method.
7. Write a program to compute $\int_{0}^{\frac{\pi}{2}} \sin x d x$ by using Simpson's $\frac{1}{3}$ rule with 200 sub intervals.
8. Evaluate the integral $\int_{0.4}^{1.6} \frac{x}{\sin x} d x$ by weddle's rule by taking 120 sub-intervals.
9. Given $y^{\prime}=3 x+y^{2}, y(1)=1.2, h=0.1$. Find $y(1.8) \mathrm{R}-\mathrm{K}$ method of four order.
10. Write a program to find a root of the equation $x \sin x-1=0$ by secant method.
11. Using iterative formula to compute $\sqrt[7]{125}$. Correct to five significant digits.
12. Find a real root of the equation $\log x=\cos x$ uisng Regula-falsi method. Correct to three significant figures.
13. Fit a linear curve to the data
X
4
6
8
10
12
$y$
13.72
12.90
12.01
11.14
10.31
14. If the prescribed curve be $f(x)=a+\beta x+\gamma x^{2}$, estimate $\alpha, \beta$ and $\gamma$ by least square method from the following data.
X
2
4
6
8
10
$y$
3.97
12.85
31.47
37.38
91.29
15. Write a program to compute $\int_{1}^{2} \sqrt{\frac{x^{2}-1}{x}} d x$ by using Simpson's $\frac{1}{6}$ rule using 1000 subintervals.

## Group - B

Answer any one from the following questions :
Each question carries 10 marks.
16. Evaluate $\int_{0}^{0.5} e^{x} d x$ by five-point Gaussian quadrature.
17. Solve the following system of linear equations by LU decomposition method :
$x+y+z=1, \quad 4 x+3 y-z=6, \quad 3 x+5 y+3 x=4$
18. Apply Newton's backward difference formula to obtain the value of $y$ at $x=1.2$ using the following table.

| $X$ | 0 | 1 | 2 | 3 | 4 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $f(x)$ | 1 | 1.5 | 2.2 | 3.1 | 4.3 |

19. Use Lagrange's interpolation formula to find $f(x)$ when $\mathrm{x}=0$ from the following table

| $X$ | -1 | -2 | 2 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| $f(x)$ | -1 | -9 | 11 | 69 |

20. Solve the following system of equations by Gaussian elimination method.
$3 x+2 y+z=10, \quad 2 x+3 y+2 z=14, \quad x+2 y+3 x=14$
21. Solve the following by Eyler's modified method.
$\frac{d y}{d x}=\log (x+y), y(0)=2$, at $x=1.4$ with $h=0.2$
22. Solve the following system by Gauss Seidal method.
$20 x+5 y-2 z=14,3 x+10 y+z=17, x-4 y+10 z=23$
23. Solve the following systems of equation by Gauss-Jacobi's iteration mehtod.
$4 x+0.24 y+0.08 z=8,0.09 x+3 y-0.15 z=9,0.04 x-0.08 y+4 z=20$
24. Find by power method, the numerically largest eigen value and the corresponding eigen vector of the following matrix :

$$
\left[\begin{array}{ccc}
1 & 3 & 2 \\
-1 & 0 & 2 \\
3 & 4 & 5
\end{array}\right]
$$

25. Find the value of $e^{x}$ when $x=0.612$ using Newton's forward difference method.

| $X$ | 0.61 | 0.62 | 0.63 | 0.64 | 0.65 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $f(x)$ | 1.840431 | 1.858928 | 1.877610 | 1.896481 | 1.915541 |

26. The distance (d) that a car has travelled at time $(\mathrm{t})$ is given below :

| Time (t) | 0 | 2 | 0.63 | 0.64 | 0.65 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Distance (d) | 0 | 40 | 160 | 300 | 380 |

27. Evaluate $y(0.02)$ given $y^{\prime}=x^{2}+y, y(0)=1$ by modified Euler's method.
28. Write a program to find the value of $y(0.1)$ from the differential equation $\frac{d y}{d x}=x+y+100, x(0)=1.2$ by fourth order Runge Kutta method.
29. If $f(0)=1, f(0.1)=0.9975, f(0.2)=0.9900, f(0.3)=0.9800$ and hence find $f(0.05)$ using Newton's forward formula.
30. Given $\log _{10} 654=2.8156, \log _{10} 658=2, .8182, \log _{10} 659=2.8189, \log _{10} 661=2.8202$, find $\log _{10} 656$ using Newton's forward formula.

# 3rd Semester Examination MATHEMATICS (Honours) 

# Paper : C 7-T <br> (Numerical Methods) 

[CBCS]

## Full Marks : 40

Time : Two Hours
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer any five questions:
(a) Compute the value of $\cos \frac{\pi}{3}$ by Taylor's series approximation of order 3 about $x=0$ and obtain the absolute error.
(b) Define truncation and round-off error in numerical calculations with example.
'c) What are the advantages and disadvantages for Secant method?
(d) Compute the value of $\sqrt{2}$ correct up to three significant figure using Newton Raphson method.
(e) If $f(1)=3, f(2)=7, f(3)=13$ then find the value of $f^{\prime}(1)$.
(f) Find the value of the integral $\int_{0}^{1} \frac{\ln (1+x)}{x} d x$ with step length 0.5 by Simpson's $1 / 3$ rule.
(g) Show that $\Delta \nabla=\Delta-\nabla$.
(h) Let $f(x)=3 x^{3}+13 x-114$. What is the value of absolute error for $\int_{0}^{1} f(x) d x$ using Simpson's $1 / 3$ rule.
2. Answer any four from the following: $\quad 5 \times 4=20$
(a) Compute $y(1.2)$ from $\frac{d y}{d x}=x^{2}+y^{2}$ with $y(1)=0$ using Runge Kutta method of $4^{\text {dh }}$ order.
(b) Determine the largest eigen value of the matrix given as follows using power method :

$$
A=\left(\begin{array}{ccc}
1 & 3 & -1 \\
3 & 2 & 4 \\
-1 & 4 & 10
\end{array}\right)
$$

(c) Derive the Newton-Cote's integration formula for a given function $y=f(x)$ in the interval $[a, b]$ with error term.
(d) Find the real root of $x^{3}-x-1=0$ using Regula N Falsi Method.
(c) Discuss Gauss Jacobi iteration Scheme for solving the system of linear equations with the sufficient conditions of convergent.
(f) Show that the rate of convergent of Newton Raphson Method for finding the real root of an equation is quadric.
3. Answer any one from the following :

$$
10 \times 1=10
$$

(d) Solve the following system of equations by LU decomposition method :

$$
2 x-3 y+4 z=8 ; x+y+4 z=15 ; 3 x+4 y-z=8
$$

(b) Discuss the Newton's Forward interpolation formula and using it find a polynomial which take the following values :

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $y$ | 41 | 43 | 47 | 53 | 61 | 71 |

# 3rd Semester Examination MATHEMATICS (Honours) 

Paper: C 5-T<br>(Theory of Real Functions and Introduction to Metric Space)

## [CBCS]

Full Marks : 60
Time : Three Hours
The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.
Group - A

1. Answer any ten questions :
(a) Prove that, $\frac{\tan x}{x}>\frac{x}{\sin x}, 0<x<\frac{\pi}{2}$
(b) Use Mean value theorem to show that $0<\frac{1}{x} \log \left(\frac{e^{x}-1}{x}\right)<1$
(()) Give the geometrical interpretation of Mean Value Theorem.
(d) Does there exist a function $\varphi$ such that $\varphi^{\prime}(x)=f(x)$ on $[0,2]$, when $f(x)=x-[x]$, where $[x]$ is the greatest integer function?

$$
(2)
$$

(e) Let $I \in \mathbb{R}$ be an interval and a function $f: I \rightarrow \mathbb{R}$ be differentiable at $C \in I$. Then if $f^{\prime}(c)>0$, prove that $f$ is increasing at $c$.
( $(t)$ Evaluate $\lim _{x \rightarrow 3}\left([x]-\left[\frac{x}{3}\right]\right)$
(g) Let $f(x)= \begin{cases}1, & x \in Q \\ 0, & x \in \mathbb{R}-Q\end{cases}$

Prove that $f$ is discontinuous at every point $c$ in $\mathbb{R}$.
(h) Define the Lipschitz's function.
(1) Find the value of limit : $\lim _{x \rightarrow \alpha}\left(1+\frac{3}{x}\right)^{x}$
(j) Let $d_{1}$ and $d_{2}$ are two metrics on a non-empty set $A$. Prove that $d_{1}+d_{2}$ is also a metric on $A$.
(k) Let $(M, d)$ be a metric space. Then prove that $\forall A, B \in M, A \subset B \Rightarrow \delta(A) \geq \delta(B)$.
(1) Let $(A, d)$ be a metric space. Then prove that $(A, \sqrt{d})$ is also a metric space.
(m) Prove that in any discrete metric space all the sets are closed.
(ni) Let $X$ be a non-empty set and $f: X \rightarrow \mathbb{R}$ be an injective function. Then prove that $d(x, y)=|f(x)-f(y)| \forall x, y \in X$ defines a metric on $X$.
(o) Show that $\lim _{x \rightarrow \infty} \frac{[x]}{x}=1$, where $[x]$ denotes the greatest integer contained in $x$ not greater than $x$.

## Group - B

2. Answer any four questions :
*a) Let $[a, b]$ be a bounded closed Interval and I denoted the set of all Riemann-Integrable ( $\mathbb{R I}$ ) function over $[a, b]$. Then prove that

$$
d(f, g)=\int_{a}^{b}|f(x)-g(x)| d x, \forall f, g \in \mathbb{R} I
$$

is a pseudo metric but not metric.
(b) Prove that $f(x)=\sin x^{2}$ is not uniformly continuous on $[0, \infty)$ but $f(x)=\sin x$ is uniformly continuous on $[0, \infty)$.
(e) State \& prove the Hausdorff property.
(d) Let $D \subset \mathbb{R}$ and $f: D \rightarrow \mathbb{R}$ be a function. Let $c$ be a limit point of $D$ and $l \in \mathbb{R}$. Then $\lim _{x \rightarrow c} f(x)=l$ iff for every sequence $\left\{x_{n}\right\}$ in $D-\{c\}$ converging to $c$, the sequence $\left\{f\left(x_{n}\right)\right\}$ converges to $l$.
(e) State and prove the Caratheodory's theorem.
(f) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$
f(x)=\left\{\begin{array}{cc}
x^{2} \sin \frac{1}{x^{2}}, & x \neq 0 \\
0, & x=0
\end{array}\right.
$$

Show that $f$ is differentiable on $\mathbb{R}$, but $f^{\prime}$ is not continuous on $\mathbb{R}$.

## Group - C

3. Answer any two questions :
$10 \times 2=20$
(a) (i) State Maclaurin's infinite series and obtain the expansion of $(1+x)^{m}$ where $m$ is any real number other than positive integer and $|x|<1$.
(ii) Let $f: I \rightarrow \mathbb{R}$ be such that $f$ has a local extremum at an interior point $C$ of I. If $f^{\prime}(c)$ exists then $f^{\prime}(c)=0$.
(b) (i) Let $I$ be an interval and a function $f: I \rightarrow \mathbb{R}$ be uniformly continuous on $I$. Then $f$ is continuous on $I$.
(ii) Let $I=[a, b]$ be a closed and bounded interval and a function $f: I \rightarrow \mathbb{R}$ be continuous on $I$. Then $f$ is uniformly continuous and $f$ is bounded on $I$.
(c) (i) Let $I=(a, b)$ be a bounded open interval and $c \in(a, b)$. If $f: I \rightarrow \mathbb{R}$ be a monotonic function on I then $\lim _{x \rightarrow c^{-}} f(x)$ and $\lim _{x \rightarrow c^{+}} f(x)$ both exist.
(ii) A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x)=\left\{\begin{array}{ll}x, & x \in Q \\ 0, & x \in \mathbb{R}-Q\end{array}\right.$.
Show that $f$ is continuous at 0 and $f$ has a discontinuity of the 2nd kind at every other point in $\mathbb{R}$. $\quad 5+5$
(d) (i) State and prove Taylor's theorem with Cauchy's form of remainder after $n$ terms.
(ii) Give exampls of a function which is :
(a) Continuous and bounded on $\mathbb{R}$, attains its. supremum but not infimum.
(b) Continuous and bounded on $\mathbb{R}$, attains its infimum but not its supremum.

# 3rd Semester Examination MATHEMATICS (Honours) 

# Paper : C 6-T <br> (Group Theory - I) 

[CBCS]
Full Marks : 60
Time : Three Hours
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer any ten questions:
$2 \times 10=20$
(1) Define abelian group. Give example of a finite abelian group.
(ii) If $G$ is a group of even order, then prove that it has an element $a \neq e$ such that $a^{2}=e$.
(iii) Does the set of all odd integers form a group with respect to addition? Give suitable justification.
(iv) Suppose that a group contains elements $a$ and $b$ such that $O(a)=4, O(b)=2$ and $a^{3} b=b a$. Find $O(a b)$.
(v) Find the order of $\alpha \beta$, if the permutations

$$
\begin{aligned}
& \alpha=\left(\begin{array}{llllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
8 & 4 & 3 & 2 & 7 & 6 & 1 & 5
\end{array}\right) \text { and } \\
& \beta=\left(\begin{array}{lllllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 5 & 2 & 7 & 8 & 1 & 6 & 4
\end{array}\right)
\end{aligned}
$$

(vi) Show that the group $\left[\{1,2,3,4\}, X_{s}\right]$ is a cyclic group.
(vii) If $a$ and $x$ are two elements of a group $G$ such that $a x a^{-1}=b$, then find $x$. If $b^{n}=e$, then find $x^{n}$.
(viii) Let $G$ and $G^{\prime}$ be two groups and $\theta: G \rightarrow G^{\prime}$ be a homomorphism of $G$ onto $G^{\prime}$. Prove that if $G$ is cyclic, then $G^{\prime}$ is also cyclic.
(ix) Let $G$ be a group and $H$ be a subgroup of $G$. Prove that $H=h H$ if and only if $h \in H$.
( K ) Prove that every cyclic group is abelian, but converse is not true in general.
(xi)Define center of a group. If $G$ be a group of order 4, what will be its center.
(xiil) Define quotient group.
(xiii) Consider the group $G=G L(2, R)$ under multiplication and $A=\left[\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right]$. Find centralizer of $A$, i.e., $C(A)$.

## ( 3 )

(xiv)) Prove that intersection of two normal subgroups is normal.
(xv) Show that the direct product $Z_{6} \times Z_{4}$ is not cyclic group.
2. Answer any four questions :
$5 \times 4=20$
(i) Prove that the set of matrices
$A_{\alpha}=\left[\begin{array}{cc}\cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha\end{array}\right]$, where $\alpha$ is a real number, forms a group under matrix multiplication. Does it abelian group?
(ii) Define even and odd permutation. Let $\alpha$ and $\beta$ belong to $S_{n^{\prime}}$ Prove that $\beta \alpha \beta^{-1}$ and $\alpha$ both are either even or odd permutation together.
(iii) Define centralizer of an element in a group. Prove that for each $a$ in a group $G$, the centralizer of $a$ is a subgroup of $G$.
(iv) How many elements of order 9 does $Z_{3} \oplus Z_{9}$ have?
(v) Find all the homomorphism of the group $(Z,+)$ to the group $(Z,+)$.
(vi) Prove that every group of prime order is cyclic.
3. Answer any two questions :
$10 \times 2=20$
(i) (a) Let $n>1$ be a fixed integer and let
P.T.O.
$u(n)=\{m \in N \mid m<n,(m, n)=1\}$. Then prove that $\left(u(n), X_{n}\right)$ is a group. If $n=100$, then what is the order of $u(n)$ ?
(b) Define Alternating group of order $n$. Find all elements of $A_{3}$.
(ii) (a) State and prove Lagrange's theorem on groups. By using Lagrange's theorem prove that if $H$ and $K$ are subgroups whose orders are relatively prime, then show that $H \cap K=\{e\}$.
$1 / 2$ (b) How many generators are there of the cyclic group of order 8 ?
$(2+4+2)+2$
(iii) (a) Let $G$ be a finite abelian group and let $p$ be a prime number such that $p$ divides order of $G$ Then prove that $G$ has an element of order p. (Cauchy's Theorem).
(b) Prove that any group of order four is abelian.
(iiv) (a) Let $\theta: G \rightarrow G^{\prime}$ be a homomorphism of a group $G$ onto a group $G^{\prime}$. Let $K=\operatorname{ker} \theta$. Then prove that $K$ is a normal subgroup of $G$ and $\frac{G}{K} \cong G^{\prime}$. (First Isomorphism Theorem).
(b) Let $H$ be a subgroup of $G$. If $x^{2} \in H$ for all $x \in G$, then prove that $H$ is a normal subgroup of $G$ and $G / H$ is commutative. $5+5$
B.Sc/3rd Sem (H)/MATH/22(CBCS)

2022

# 3rd Semester Examination MATHEMATICS (Honours) 

Paper: SEC 1-T

[CBCS]

## Full Marks : 40

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

## (Logic and Sets)

## Group - A

Answer any five questions: $\quad 2 \times 5=10$

1. If $A=\{x \in R: 0 \leq x \leq 1\}, B=\{x \in R: 1 \leq x \leq 2\}$, then find $A \times B$.
2. Define equivalence class of an element. Give an example.
3. Define partial order relation. Give an example.
A. Define power set of a set. Give an example.
4. Define valid argument.
5. What is negation of a predicate? Give an example.
6. If $D_{1}=\{2,3,4\}, D_{2}=\{6,7,8\}$ and $D_{3}=\{3,4,5\}$ are the domains of the variables $x, y$ and $z$ respectively, then find the truth value of the following quantified predicate.
$\exists x \forall y \exists z p(x, y, z)$, where $p(x, y, z): x^{2}+y^{2}<3 z^{2}$.
7. Find the negation of the following statement.

$$
(\forall x \in D)(x+3<10) \text {, where } D=\{1,2,3,4\} .
$$

Group - B
Answer any four questions:
$5 \times 4=20$

1. Prove that $p \rightarrow q, q \rightarrow r \vDash p \rightarrow r$.
$: \sqrt{2}$. Verify the statement, by truth table :

$$
p \leftrightarrow q \equiv(p \rightarrow q) \wedge(q \rightarrow p) .
$$

3. Determine whether the following argument is correct :

If it rains in Spain, then Steffi will win and Gabriela will lose. If Gabriela loses, then Germany will become the team champion. If Steffi wins, then Jenifer will be disappointed. If it does not rain in Spain, then Monica will win. Therefore, either Jenifer will be disappointed and Germany will be the team champion or Monica will win.
4. Examine whether the following relation $\rho$ defined on the set $Z$ is an equivalence relation.

$$
\rho=\{(a, b) \in Z \times Z:|a-b| \leq 3\}
$$

## ( 3 )

5. If $\rho$ be an equivalence relation on a set $S$ and $a, b \in S$. Then prove that $c l(a)=c l(b)$ iff $a \rho b$.
6. If $A, B, C$ are subsets of a universal set $S$, then prove that $(A-B) \cup(B-C) \cup(C-A)=(A \cup B \cup C)-(A \cap B \cap C)$.

## Group-C

Answer any one question:

1. (i) Define Cartesian product on $n$ non-empty sets. If $O(A)=n$, and $O(B)=m$, then find $O(P(A \times B))$.
(ii) Let $A, B$ be subsets of a universal set $S$. Then prove that $A=B$ iff $A \Delta B=\phi$.
(iii) If $\rho$ be a relation defined on $N \times N$ such that " $(a, b) \rho(c, d)$ iff $a d=b c$ " for $(a, b),(c, d)$ $\in N \times N$, then check whether $\rho$ is an equivalence relation?
2. (i) What is universal quantifier? Symbolize the following sentences using predicates and quantifiers :
(a) Everybody is not rich.
(b) Every integer is divisible by 6 iff it is divisible by both 2 and 3 .

## ( 4 )

(ii) Test the logical validity of the following argument : Some students work hard. All males work hard. Therefore some students are male.
(iii) Write down the De Morgan's laws and absorption laws in the set of statement formulas. After then prove any one of the De Morgan's laws and the absorption laws by truth table.
$1+3$

